Table 2 $10^5\,\theta''$

	Exact		Experimental	
Run No.	Fit	Estimate	Fit	Estimate
582	−0.48	−0.50	0.96	−0.50
575	−0.87	−0.83	0.44	−0.61
584	−1.63	−1.57	−2.02	−1.61
585	2.84	2.85	6.52	−0.69

with those from Eq. (2), however, a systematic overestimate of the exact $\alpha_{\max}$ was observed for Run 585, which had the largest amplitude motion. This was felt to be caused by fitting the actual elliptic function solution with a sine wave. This bias can be eliminated through the approximation

$$sn\,u \doteq \sin y(1 + 4q\cos^2 y) \tag{6}$$

where $q = \exp(-\pi E_1' E_1^{-1})$, $E_1' = E_1(1 - k^2)^{1/2}$, $E_1 = E_1(k)$ complete elliptic integral of first kind, $k = -m(2 + m)^{-1}$, $m = 4K^2C_4C_3^{-1}$, and $u = (2E_1/\pi)y$. If $sn\,u$ is fitted by least squares to $A \sin y$, Eq. (6) yields

$$A = 1 + q \tag{7}$$

This correction for Run 585 is −2.4%. $3K^2$ and K^2 are each reduced by 4.8% in Figs. 2 and 3 and lines refitted, with the results $C_1 = -0.00602$, $C_2 = 0.000100$, $C_3 = 0.02074$, $C_4 = -0.0000380$. Thus, we see that the quasi-linear technique can yield the nonlinear damping coefficients to 5% and the cubic static moment coefficient to 2% from the exactly calculated points.

Analysis of Experimental Data

The actual flight data can now be fitted by Eqs. (2–4) with the results given in Tables 1 and 2. The standard error of the fits is quite good, but θ'' is now poorly predicted by Eq. (5). This probably means that C_3 and C_4 are not constants but must be allowed to be functions of Mach number,

$$\therefore \theta'' = [6C_4K^2\lambda + C_3' + 3C_4'K^2](2\theta')^{-1} \tag{8}$$

where $C_j' = (dC_j/dM)(dM/dx)$, M = Mach number. No provision for this effect is made in Chapman-Kirk analysis of these data although it could easily be incorporated by assuming a linear dependence of $C_{3,4}$ on Mach number and thereby increase the number of unknown coefficients of the differential equation to six.

The discrepancy shown in Fig. 1 can now be explained easily. The data contain both varying frequency and exponential damping. The fitted curve implicitly assumes these to be related by Eq. (5). Since the frequency variation dominates, the damping rate is given an erroneous bias. In Fig. 3 the quasi-linear damping is plotted for the experimental data and we see substantially different values of C_1 and C_2 are indicated. These values predict a limit cycle oscillation of amplitude 23° instead of the value of 15° implied by Ref. 1.

Figures 2 and 3 also show an additional advantage of the quasi-linear method. Namely, it gives an indication of the accuracy of the determination of the nonlinear coefficients since they appear as slopes of fitted lines. Indeed, we see that the nonlinear damping coefficient is poorly determined whereas the cubic static moment coefficient is very well determined.

Table 3 Quasi-linear results for exact input

	C_1	C_2	C_3	C_4
Exact	−0.00607	0.000105	0.02078	−0.0000375
Quasi-linear	−0.00600	0.000095	0.02072	−0.0000362
Quasi-linear[a]	−0.00602	0.000100	0.02074	−0.0000380

[a] Modified by Eq. (7).

References

[1] Chapman, G. T. and Kirk, D. B., "A New Method for Extracting Aerodynamic Coefficients from Free Flight Data," *AIAA Journal*, Vol. 8, No. 4, April 1970, pp. 753–758.

[2] Murphy, C. H., "The Prediction of Nonlinear Pitching and Yawing Motion of Symmetric Missiles," *Journal of the Aeronautical Sciences*, Vol. 24, No. 7, July 1957, pp. 473–479.

[3] Murphy, C. H., "Quasi-Linear Analysis of the Nonlinear Motion of a Nonspinning Symmetric Missile," *Journal of Applied Mathematics and Physics* (*ZAMP*), Vol. 14, Sept. 1963, pp. 630–643.

[4] Murphy, C. H., "Free Flight Motion of Symmetric Missiles," BRL Rept. 1216, AD442757, July 1963, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

[5] Chapman, G. T. and Kirk, D. B., private letter Oct. 15, 1969.

Reply by Authors to C. H. Murphy

Gary T. Chapman* and Donn B. Kirk†

NASA Ames Research Center, Moffett Field, Calif.

The quasi-linear method in the past has proven to be completely adequate in a great many instances; Dr. Murphy has certainly demonstrated its applicability in this example. However, there are problems for which the quasi-linear method would not be expected to give good results as, for example, when the test configuration departs greatly from being axisymmetric and the nonlinearities in the governing differential equations become large. Further, when using an approximate method on problems of this type, it is difficult to know a priori just how valid the solution is. Theoretical approximations interact in a variety of different ways with errors in the experimental data. The present method[1] eliminates this problem and, conceptually at least, can treat the complete differential equations involving six degrees of freedom.

Reference

[1] Chapman, G. T. and Kirk, D. B., "A Method for Extracting Aerodynamic Coefficients from Free-Flight Data," *AIAA Journal*, Vol. 8, No. 4, April 1970, pp. 753–758.

Received July 23, 1970.

* Research Scientist, Hypersonic Free-Flight Branch, Vehicle Environment Division. Member AIAA.

† Research Scientist, Hypersonic Free-Flight Branch, Vehicle Environment Division.

Reply by Author to M. N. Rao

Belur V. Dasarathy*

Computer Sciences Corporation, Huntsville, Ala.

As pointed out by M. N. Rao in his comments[1] on our earlier studies[2,3] on classes of second- and third-order nonlinear systems, there indeed exists a definite relationship between the various nonlinear terms involved in the governing differential equation which permits it to be reducible to an equivalent linear differential equation (i.e., to an integrable form). However, a more general relationship than that put

Received July 15, 1970.

* Member AIAA.